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Tunable Lyapunov exponent in inverse magnetic billiards

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The stability properties of the classical trajectories of charged particles are investigated in a two-dimensional inverse magnetic domain, where the magnetic field is zero inside the domain and constant outside. As an example, we present detailed analysis for stadium-shaped domain. In the case of infinite magnetic field, the dynamics of the system is the same as in the Bunimovich billiard, i.e., ergodic and mixing. However, for weaker magnetic fields, the phase space becomes mixed and the chaotic part gradually shrinks. The numerical measurements of the Lyapunov exponent (based on the technique of Jacobi fields) and the regular-to-chaotic phase space volume ratio show that both quantities can smoothly be tuned by varying the external magnetic field. A possible experimental realization of the inverse magnetic billiard is also discussed.

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The investigation of standard billiard models (e.g., Bunimovich [1] and Sinai billiards [2]) has played a pioneering role since the very beginning of chaos theory. Recent developments in nanotechnology have made it possible to experimentally realize such systems by electrostatically confining a two-dimensional electron gas (2DEG) in high mobility heterostructures [3,4]. In these systems, the dynamics of the electrons is dominated by ballistic motion. In the past decade, a new perspective of the research of semiconductor systems has emerged by the application of spatially inhomogeneous magnetic fields. The inhomogeneity of the magnetic field can be realized experimentally either by varying the topography of the electron gas [5], or using ferromagnetic materials [6], or depositing a superconductor on top of the 2DEG [7]. Numerous theoretical works also show the increasing interest in the study of electron motion in inhomogeneous magnetic field [8].

The aim of our theoretical work is to present an alternative two-dimensional billiardlike system which exhibits a crossover between a well-known, ergodic and mixing billiard system, and a pathological integrable system, as the applied magnetic field is changed. The magnetic field is inhomogeneous: zero in a compact region of the plane and nonzero outside. We suppose that the system is in the ballistic regime, such as in many other works (see, e.g. Refs. [3,9]), and our treatment is purely classical. Two characteristic quantities of the dynamics of this type of system, which we call *inverse magnetic billiard*, are calculated numerically as a function of the external magnetic field β: the Lyapunov exponent $\lambda(\beta)$ (of the dominating chaotic component), and the regular-to-chaotic phase space volume ratio $\varrho(\beta)$. The obtained numerical results show that both quantities are smooth functions of the magnetic field, which means that the global dynamics of the system passes continuously from the integrable ($\beta=0$) to the fully chaotic case ($\beta=\infty$). As we shall see below, there is also a clearly visible correlated dependence between the variation of the quantities $\lambda(\beta)$ and $\varrho(\beta)$. These results imply that *the degree of chaoticity can smoothly be tuned* by the external magnetic field. We note that Kosztin *et al.* have made similar investigations and observations in Andreev billiard systems [10].

To demonstrate our general arguments mentioned above, we choose a well-known chaotic system, the Bunimovich billiard shown in Fig. 1. The magnetic field (perpendicular to the plane) is zero inside the stadium-shaped region and constant β outside. A part of a typical classical trajectory is depicted in Fig. 1, for an intermediate value of the magnetic field $\beta=2$. The trajectories in the configuration space are straight segments inside the stadium, and circular arcs of cyclotron radius $R_c=1/\beta$ out of this domain. (We assume, for simplicity, that the particle has unit mass, charge, and speed.) At the boundary of the domain, the two pieces of the trajectory join tangentially. As the magnetic field tends to infinity, $\beta\to\infty$, the charged particle spends less and less time outside the stadium, and it is also easy to see that in the limiting case its motion is described by an elastic reflection from the wall. For this reason we call our system inverse magnetic billiard, although in the case of finite field no real scatterings take place at the boundaries.

According to the result of Bunimovich [1], the stadium-shaped inverse magnetic billiard system is ergodic and mixing in the $\beta=\infty$ case, but as the magnetic field is decreased, the dynamics becomes partially regular and gradually more and more phase space volume is occupied by the Kolmogorov-Arnold-Moser tori (or $\mathbb{R}^2$ leaves in the $\beta=0$ case), which means that the phase space is mixed. This phenomenon can clearly be observed on the Poincaré sections (see Fig. 2) made for different magnetic field values. The individual points in the Poincaré sections are plotted each

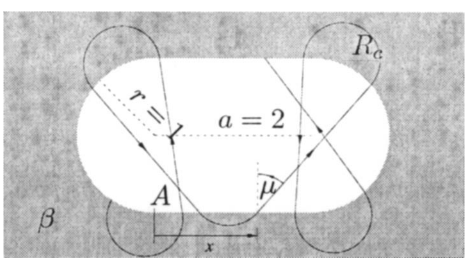

FIG. 1. The trajectories of a charged particle in the inverse magnetic billiard. The cyclotron radius is $R_c=1/\beta=1/2$, in dimensionless units.

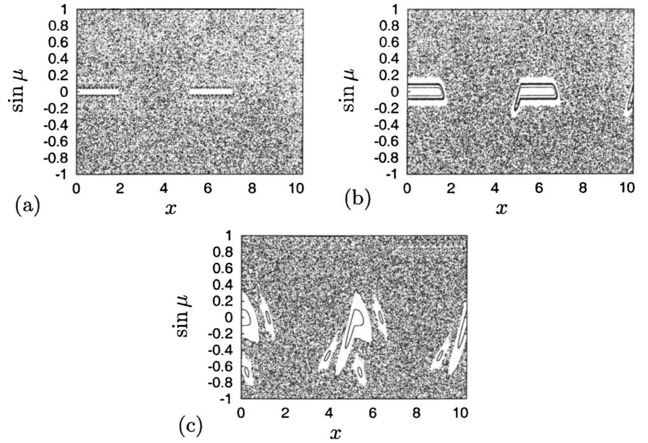

FIG. 2. The Poincaré section of the phase space. The points in the dominating chaotic region were obtained by 50 000 iterations of a single trajectory, while for depicting the islands corresponding to the regular regions, a few different initial conditions were used. The values of the cyclotron radii are $R_c=0.05$, $R_c=0.3$, $R_c=1$, respectively.

time the particle enters the zero magnetic field region and crosses the boundary of the stadium. The *x* coordinate of the points $(0\leqslant x<4+2\pi)$ gives the position of the crossing, measured in counterclockwise direction from the point *A* along the perimeter of the stadium, while the *y* coordinate of the points $(-1\leqslant y\leqslant 1)$ denotes the sine of the angle μ representing the direction of the trajectory, relative to the normal of the boundary (see Fig. 1). This Poincaré section represents only the relevant part of the phase space, i.e., the trajectories intersecting the stadium region. It is well known that in this parameter space, the Poincaré map is area preserving [11].

It is evident from Fig. 2 that for high magnetic fields, the system is (almost) completely chaotic but with decreasing magnetic field, the volume of the regular regions gradually increases. As we have seen before, for $\beta=\infty$ the system is identical to the Bunimovich billiard, however, in the $\beta\rightarrow 0$ limit the system becomes pathological in the sense that the cyclotron radius tends to infinity, so the particle returns to the stadium domain after longer and longer time intervals.

In order to quantitatively characterize this change of the phase space portrait, we have numerically investigated the regular-to-chaotic phase space volume ratio ϱ as a function of the cyclotron radius $R_c=1/\beta$ (i.e., the inverse magnetic field), and the results are shown in Fig. 3. The function $\varrho(R_c)$, measured by the box-counting method with a grid of 250×250 rectangular sites, is smooth, and its behavior is characteristically different for higher and lower magnetic fields. For cyclotron radii less than $R_1\approx 0.01$ (i.e., for magnetic fields larger than $\beta_1\approx 100$), the system is dominantly chaotic, and the area of the regular phase space regions is practically negligible [see also Fig. 2(a)]. For cyclotron radii larger than $R_2\approx 0.3$, however, the regular part increases on the Poincaré section [see also Fig. 2(c)]. Between these two extremities, i.e., for cyclotron radii comparable to the characteristic size of the billiard, the phase space of the system is definitely mixed [Fig. 2(b)] with regular islands of considerable area.

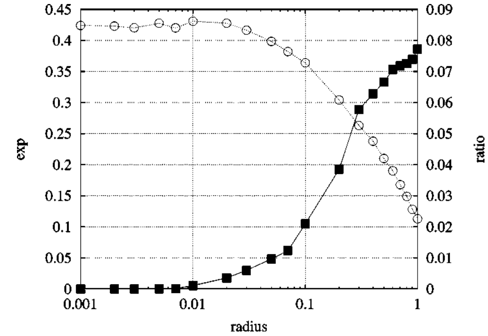

FIG. 3. The regular to chaotic phase space volume ratio $\varrho(R_c)$ (full squares) and the Lyapunov exponent $\lambda(R_c)$ (open circles) as a function of $R_c=1/\beta$.

Although the volume of the chaotic bands inside the regular islands (ignored in our treatment) is nonzero in principle, the numerical simulations demonstrate (see Fig. 2) that their contribution to the chaotic phase space volume is negligible for this system.

The positivity of the Lyapunov exponent $\lambda(R_c)$ is one of the most characteristic features of magnetic or nonmagnetic billiard systems (see, e.g., Ref. [12] and references therein). We have numerically computed $\lambda(R_c)$ of the dominating chaotic component as a function of the cyclotron radius R_c (see Fig. 3).

The obtained function $\lambda(R_c)$ is again smooth, as $\varrho(R_c)$. It is also clearly visible that the numerical value of the Lyapunov exponent strongly correlates with the regular phase space ratio $\varrho(R_c)$ measured previously. For weak magnetic fields (if $\beta\lesssim 2$), the Lyapunov exponent is also small, but as the magnetic field grows, the value of λ increases, too, and for strong fields (if $\beta\gtrsim 100$) it saturates at the value $\lambda_\infty\approx 0.43$, which agrees exactly with the Lyapunov exponent of the ordinary Bunimovich billiard [13].

In order to measure the Lyapunov exponent, we have investigated the infinitesimal variations of the trajectories with the method of Jacobi fields, which was originally developed for the stability analysis of the geodetic flow on curved Riemannian manifolds [14]. The method has successfully been applied to magnetic billiard systems on planar [15] as well as curved surfaces [16,17]. The main idea of the method is to study the evolution of the so-called Jacobi fields along a particular trajectory in the configuration space, which describe the infinitesimal variations of the trajectory. This technique is essentially the same as the method using the tangent map [11], but our approach is more transparent. The basic technical importance is that in our investigations, the coordinates describing the infinitesimal variations are chosen in a more natural way: they are related to the unvaried trajectory itself, and not to the somewhat artificial parameters of the space of the Poincaré section. As a result, the stability matrices (i.e., the tangent maps) have a much simpler form.

In more details, let $\gamma_0(t)$ denote the trajectory in the configuration space $\mathcal{M}$, whose stability properties we intend to investigate, and let $\gamma_\varepsilon(t)$ be a one-parameter family of varied trajectories around the unvaried one γ_0, i.e., for all $\varepsilon\in(-\varepsilon_0,\varepsilon_0)$, $\varepsilon_0>0$ the curve γ_ε is a real trajectory in the configuration space, $\gamma_{\varepsilon=0}=\gamma_0$, and the map $\gamma:(-\varepsilon_0,\varepsilon_0)$

$\times\mathbb{R}\to\mathcal{M}$, $(\varepsilon,t)\mapsto\gamma_\varepsilon(t)$ is everywhere continuous, and piecewise smooth. (It is not smooth at the boundary of the "billiard.") The *Jacobi field* or *infinitesimal variation vector field* V_{γ_0} corresponding to the variation γ_ε is the partial derivative $V_{\gamma_0}(t)=\partial\gamma_\varepsilon(t)/\partial\varepsilon|_{\varepsilon=0}$.

It can be shown that the Jacobi fields $V_{\gamma_0}(t)$ satisfy certain second-order differential equation, called *Jacobi equation*; it is due to the fact that the varied curves γ_ε are also real trajectories [14,16]. In two-dimensional billiard systems, we found it convenient to fix the base vectors $\{\dot\gamma_0(t),\dot\gamma_0^\perp(t)\}$ of the coordinate system to the investigated trajectory $\gamma_0(t)$, in such a way that $\dot\gamma_0(t)$ is the (unit) vector tangential to the trajectory at the time instant t, and $\dot\gamma_0^\perp(t)$ is obtained from $\dot\gamma_0(t)$ by a rotation through $+90°$. In this basis, the Jacobi field is written as $V_{\gamma_0}(t)=\xi(t)\dot\gamma_0(t)+\eta(t)\dot\gamma_0^\perp(t)$, and for characterizing a given infinitesimal variation the initial conditions $\xi(t_0)$, $\eta(t_0)$, $\dot\xi(t_0)$, and $\dot\eta(t_0)$ have to be given. (The real functions ξ and η are the coordinates of the Jacobi field V_{γ_0}.)

The number of these initial data can further be reduced by 2, if we notice that (i) the longitudinal variations $\xi(t)$ as well as (ii) the variations altering the speed (i.e., for which $\dot\xi-\beta\eta\neq0$, see, e.g., Ref. [16]) are irrelevant in the present investigation, and they decouple from the other coordinates, so they can be disregarded. [In the case (i) the Jacobi field is tangential to the unvaried trajectory γ_0, thus the varied curves are just the time shifts of the original one, while (ii) means that we restrict the attention to a constant energy shell of the phase space, as it is usual in Hamiltonian systems.]

In planar billiard systems, it is an elementary geometric problem to find the solutions of the Jacobi equation in terms of the transverse coordinates $\eta(t)$ and $\dot\eta(t)$ (see, e.g., Ref. [15]). Generally, the solution is given by a linear transformation $\left[\begin{smallmatrix}\eta'\\ \dot\eta'\end{smallmatrix}\right]=\mathbf{L}\left[\begin{smallmatrix}\eta\\ \dot\eta\end{smallmatrix}\right]$, where the matrix $\mathbf{L}$ has the following special forms for the straight flight in zero magnetic field ($\mathbf{P}$), for the curved flight in nonzero magnetic field ($\mathbf{E}$), and for the boundary transition ($\mathbf{T}$) with magnetic field change $\Delta\beta$, respectively:

$$\mathbf{P}(t)=\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix},\quad \mathbf{E}(t,\beta)=\begin{bmatrix}\cos(\beta t) & \frac{1}{\beta}\sin(\beta t)\\ -\beta\sin(\beta t) & \cos(\beta t)\end{bmatrix}, \tag{1}$$

$$\mathbf{T}(\Delta\beta,\mu)=\begin{bmatrix}1 & 0\\ \Delta\beta\tan\mu & 1\end{bmatrix}.$$

Here t is the time of flight (so βt is the angle of flight), β denotes the magnetic field, and μ is the angle of incidence at the boundary, measured in the way shown in Fig. 1. It is worth noticing that all the three types of matrices are one-parameter subgroups of SL(2,$\mathbb{R}$), i.e., of the group of 2×2 real matrices with unit determinant. The matrices $\mathbf{P}$ and $\mathbf{T}$ are parabolic, while the transformations $\mathbf{E}$ are elliptic.

For investigating the long time stability of a given trajectory γ_0, the eigenvalues (or the trace) of the product matrix

$$\cdots(\mathbf{T}_3'\mathbf{E}_3\mathbf{T}_3\mathbf{P}_3)(\mathbf{T}_2'\mathbf{E}_2\mathbf{T}_2\mathbf{P}_2)(\mathbf{T}_1'\mathbf{E}_1\mathbf{T}_1\mathbf{P}_1) \tag{2}$$

have to be calculated, where the individual matrices in the expression describe, in reverse order, the stability of the corresponding segments of the motion (in the billiard, through the boundary outwards, in the magnetic field and back again into the billiard through the boundary). This group of four matrices corresponds to a cycle in the Poincaré sections of Fig. 2. (The matrices $\mathbf{T}$, $\mathbf{T}'$ correspond to the outward and the inward passage through the boundary, respectively.)

In our simulations, matrices (1) and product (2) corresponding to about 25 000 cycles were calculated explicitly, and the Lyapunov exponents shown in Fig. 3 were computed as the logarithm of the largest eigenvalue (practically, the trace) of the resulting matrix divided by the total time of flight.

The fact that in the $\beta\to\infty$ limit the inverse magnetic billiard gives back the dynamics of the normal billiard system with elastic walls can be checked also in terms of the stability matrices. A bit lengthy but straightforward calculation yields that if the billiard wall is a circle of curvature q, then

$$\lim_{\beta\to\infty}[\mathbf{T}(-\beta,-\mu)\mathbf{E}(t,\beta)\mathbf{T}(\beta,\mu)]=-\begin{bmatrix}1 & 0\\ -\dfrac{2q}{\cos\mu} & 1\end{bmatrix}, \tag{3}$$

which is the stability matrix corresponding to an elastic reflection on the wall of curvature q [16], as it is expected.

We now comment on the conditions of the experimental realization of the inverse magnetic billiards. This arrangement can be realized by depositing a superconductor patch (e.g., of stadium shape) on the top of a 2DEG (e.g., using GaAs/Al$_x$Ga$_{1-x}$As heterostructure) and applying an external homogeneous magnetic field. The magnetic field is excluded from the region covered by the superconductor, due to the Meissner effect.

There are four characteristic lengths in the system: the Fermi wavelength (typically $\lambda_F=40$ nm [3]), the characteristic length r of the system (e.g., the radius of the stadium), the cyclotron radius R_c, and the mean free path l (which can be as high as 10^4 nm [3]). The classical ballistic motion of the electrons requires that $\lambda_F\ll r,R_c\ll l$. (The last condition assures that the electron travels through several Poincaré cycles without scattering on impurities.) Figure 3 shows that the relevant values of the ratio R_c/r are in the range of $0.01-1.0$. The magnetic field can be as high as a few tesla without destroying superconductivity. This implies that $R_c\gtrsim50$ nm (using that the effective mass of electrons $m_{\mathrm{eff}}=0.067m_e$, where m_e is the mass of the electron, and $E_F=14$ meV [3]). Assuming that the size of a superconductor grain is about $r=1$ μm, the cyclotron radii are 50, 300, 1000 nm corresponding to data R_c/r in Fig. 2. This implies that parameter β in Fig. 2 corresponds to the experimental values of the magnetic field 2, 0.3, 0.1 T, respectively. The semiclassical or full quantum mechanical treatment of the problem can be an extension of our work.

The advantage of our suggested setup in comparison with Andreev billiards (in which chaoticity can also be tunable) is that in our system, the electrons travel in a homogeneous heterostructure without any scattering on the boundary of the stadium, whereas in the case of Andreev billiards the usually non-negligible normal reflections at the interface of the normal and superconducting region may suppress the effect as discussed in Ref. [10].

We would like to stress that in the inverse magnetic billiards *the degree of chaoticity can smoothly be tuned* by varying only one experimental parameter, namely the external magnetic field. This may motivate the experimental realization and study of our presently proposed system.

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[1] L. Bunimovich, Funct. Anal. Appl. **8**, 73 (1974); L.A. Bunimovich, Commun. Math. Phys. **65**, 295 (1979); **130**, 599 (1990).

[2] Ya.G. Sinai, Russ. Math. Surveys **25**, 137 (1970).

[3] C.W.J. Beenakker and H. van Houten, Solid State Phys. **44**, 1 (1991).

[4] C.M. Marcus, A.J. Rimberg, R.M. Westervelt, P.F. Hopkins, and A.C. Gossard, Phys. Rev. Lett. **69**, 506 (1992); D. Weiss, M.L. Roukes, A. Menschig, P. Grambow, K. von Klitzing, and G. Weimann, *ibid.* **66**, 2790 (1991).

[5] C.L. Foden, M.L. Leadbeater, J.H. Burroughes, and M. Pepper, J. Phys.: Condens. Matter **6**, L127 (1994); M.L. Leadbeater *et al.*, Phys. Rev. B **52**, R8629 (1995).

[6] M.L. Leadbeater *et al.*, J. Appl. Phys. **69**, 4689 (1991); K.M. Krishnan, Appl. Phys. Lett. **61**, 2365 (1992).

[7] A.K. Geim *et al.*, Nature (London) **390**, 259 (1997); A. Smith *et al.*, Phys. Rev. B **50**, 14 726 (1994).

[8] J.E. Müller, Phys. Rev. Lett. **68**, 385 (1992); F.M. Peeters and P. Vasilopoulos, Phys. Rev. B **47**, 1466 (1993); M. Calvo, *ibid.* **48**, 2365 (1993); F.M. Peeters and A. Matulis, *ibid.* **48**, 15 166 (1993); A. Matulis and F.M. Peeters, Phys. Rev. Lett. **72**, 1518 (1994); I.S. Ibrahim and F.M. Peeters, Phys. Rev. B **52**, 17 321 (1995); H.-S. Sim *et al.*, Phys. Rev. Lett. **80**, 1501 (1998); D. Urbach, diplome thesis, Lausanne, Switzerland, 1998; N. Kim, G. Ihm, H.-S. Sim, and K.J. Chang, Phys. Rev. B **60**, 8767 (1999); N. Kim, G. Ihm, H.-S. Sim, and T.W. Wang, *ibid.* **63**, 235317 (2001); S.M. Badalyan and F.M. Peeters, *ibid.* **64**, 155303 (2001); M. Governale and D. Boese, Appl. Phys. Lett. **77**, 3215 (2000); J. Reijniers, F.M. Peeters, and A. Matulis, Phys. Rev. B **64**, 245314 (2001).

[9] C.W.J. Beenakker and H. van Houten, Phys. Rev. Lett. **63**, 1857 (1989); R. Fleischmann, T. Geisel, and R. Ketzmerick, *ibid.* **68**, 1367 (1992).

[10] I. Kosztin, D.L. Maslov, and P.M. Goldbart, Phys. Rev. Lett. **75**, 1735 (1995).

[11] M.V. Berry, Eur. J. Phys. **2**, 91 (1981); O. Meplan, F. Brut, and C. Gignoux, J. Phys. A **26**, 237 (1993).

[12] B. Gutkin, U. Smilansky, and E. Gutkin, Commun. Math. Phys. **208**, 65 (1999); B. Gutkin, *ibid.* **217**, 33 (2001).

[13] G. Benettin and J.M. Strelcyn, Phys. Rev. A **17**, 773 (1978); O. Biham and M. Kvale, *ibid.* **46**, 6334 (1992).

[14] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience Tracts in Pure and Applied Mathematics, Vol. 15 (Interscience, New York, 1969).

[15] T. Tasnádi, Commun. Math. Phys. **187**, 597 (1997).

[16] T. Tasnádi, J. Math. Phys. **37**, 5577 (1996).

[17] T. Tasnádi, J. Math. Phys. **39**, 3783 (1998).